Basics of Mathematical Optimization

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A mathematical optimization problem has either of the following forms:

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & x \in X \end{aligned} \qquad \text{or} \qquad \begin{aligned} \max \quad & f(x) \\ \text{s.t.} \quad & x \in X. \end{aligned} \tag{1}$$

Here, X is a set of variables x, while f is a function defined on X such that we can compare its values (partially ordered). Let $\mathbb{R}$ denote the real numbers. In this course, and in many *real*-world problems, the set X is a subset of some n-dimensional real vector space $X \subseteq \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ takes value in real numbers. We call the set X the *feasible region* of the problem, the variables x the *decision variables*, and the function f the *objective function*. Any $x \in \mathbb{R}^n$ is *feasible* if $x \in X$ and *infeasible* otherwise. When $X = \varnothing$, we say that the problem is *infeasible*.

The minimum or maximum value of the objective function is called the *optimal value* of the optimization problem, if it exists (Example 1). Certain values of the decision variables $x^* \in \mathbb{R}^n$ are called *an optimal solution* if $f(x^*) = \min_{x\in X} f(x)$ or $f(x^*) = \max_{x\in X} f(x)$, and denoted as $x^* \in \arg\min_{x\in X} f(x)$ or $x^* \in \arg\max_{x\in X} f(x)$. Despite some notational difference, we do not really need to develop different theories for the two forms because we can transform $\max_{x\in X} f(x)$ into $\min_{x\in X} -f(x)$, where only the sign of the optimal value is changed and the set of optimal solutions remains unchanged. For now, we only discuss minimization problems for notational convenience.

An optimization problem does not necessarily have any optimal value or optimal solutions. When it does, it may not have a unique optimal solution. Thus to be rigorous, one would only say "the" optimal solution when it exists and is known to be unique.

Example 1.

- *Let $X = \mathbb{R}$ and $f(x) = x$. For any $x \in \mathbb{R}$, there is a real number $a < x$. Therefore, f does not have a minimum on X. This also implies that the optimization problems $\min_{x\in X} f(x)$ do not have optimal solutions.*
- *Let $X = \{x \in \mathbb{R} : 0 < x < 1\} \subseteq \mathbb{R}$ and $f(x) = x$. For any $x \in X$, notice that*

$a := x/2 \in X$. Clearly $f(a) < f(x)$ so f does not have a minimum on X. This also implies that the optimization problems $\min_{x\in X} f(x)$ *do not have optimal solutions.*

- *Let $X = \{x \in \mathbb{R} : x \geq 1\} \subseteq \mathbb{R}$ and $f(x) = 1/x$. Notice that $f(x) > 0$ for any $x \in X$, and for any $a > 0$, we can find $x = 1 + 1/a \in X$ such that*

$$f(x) = \frac{1}{x} = \frac{a}{a+1} < a.$$

  In plain words, no matter how small a positive number a is, we can always find a decision variable x such that $f(x) < a$. Therefore, the optimization problem $\min_{x\in X} f(x)$ *does not have optimal solutions.*
- *Let $X = \mathbb{R}$ and $f(x) = 0$ (i.e., a constant function). Any number $x \in \mathbb{R}$ is an optimal solution to both the minimization problem, and therefore, the optimization problem does not have a unique solution.*

We say a minimization problem is *bounded* if we can find a real number $a \in \mathbb{R}$ such that $f(x) > a$ for all feasible decision variables $x \in X$, and *unbounded* otherwise. When the optimal value is not guaranteed to exist, some people write inf instead of min to denote the largest such lower bound, and use the convention that the optimal value of an unbounded minimization problem is $-\infty$. Similarly, the smallest upper bound of a maximization problem is sometimes denoted as sup and the optimal value of an unbounded maximization problem is $+\infty$. With this convention, an infeasible minimization (resp. maximization) problem has its optimal value $+\infty$ (resp. $-\infty$). Please note that the infinity notation is not a real number and should be treated with care. Whenever a problem is unbounded or infeasible, there is no optimal solution, $\arg\min_{x\in X} f(x) = \varnothing$.

When the optimization problem is bounded, the feasible region is *closed*, and the objective function is *continuous*, then the existence of the optimal value and optimal solutions is guaranteed. We do not define continuous functions in this class, but the common elementary functions (e.g. linear, polynomial, rational power, exponential, trigonometric, and their sums, products, inverses, compositions) are all continuous functions inside their domains. Using these functions in *nonstrict* inequalities would automatically define a closed feasible region. Requiring some of the variables to be integers also leads to a closed feasible region.

In this course, we will mostly consider the case where the feasible region X consists of discrete or continuous values, and is defined functionally by a finite number of equalities and inequalities. That is, given an index $n' \leq n$ and functions $g_1, \ldots, g_m : \mathbb{R}^n \to \mathbb{R}$,

$$X := \{x \in \mathbb{Z}^{n'} \times \mathbb{R}^{n-n'} : g_i(x) \leq 0, i = 1, \ldots, m', \; g_j(x) = 0, j = m'+1, \ldots, m\}. \quad (2)$$

Each of the equalities or inequalities is called a *(functional) constraint* on our decision variables x. We are using the convention $g_i(x) \leq 0$ for $i = 1, \ldots, m'$ because any inequality $g'(x) \geq 0$ can be equivalently rewritten as $-g'(x) \leq 0$, and only considering nonstrict inequalities out of the concern of optimal solution existence, as discussed above. The optimization problem (1) with a feasible region (2) can be more directly written as

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g_i(x) \leq 0, \quad i = 1, \ldots, m', \\ & g_j(x) = 0, \quad j = m'+1, \ldots, m, \\ & x \in \mathbb{Z}^{n'} \times \mathbb{R}^{n-n'} \end{aligned} \tag{3}$$

without the need to explicitly specify the set X. With any functional constraint (3), i.e., $m > 0$, the problem is called *constrained* optimization, and *unconstrained* otherwise. We say that the problem (3) is *linear* if $f, g_1, \ldots, g_m$ are all affine linear functions, and *nonlinear* otherwise. When all of the variables must take integer values, i.e., $n' = n$, we say that the problem (3) is *discrete* or *integer* optimization; if all of the variables can take continuous values, i.e., $n' = 0$, then the problem is often called *continuous* optimization; and in the case $0 < n' < n$ we say that the problem is *mixed-integer* optimization.

For constrained optimization modeling, we can follow the procedure below.

(i) Describe the relevant data of the problem.

(ii) Identify and describe the decision variables.

(iii) Describe the sign, bounds, and type restrictions on individual variables.

(iv) Write the constraints in terms of the decision variables.
  - If any additional variable is needed, go back to Step (ii).

(v) Write the objective function in terms of the decision variables.
  - If any additional variable is needed, go back to Step (ii).

Example 2. *One wants to design a aluminum can in the shape of a cylinder with height h and radius r with the minimum usage of aluminum (Figure 1), such that the following requirements are satisfied.*

- *The height h must be at least three times as large as the radius r.*
- *The height h can be at most four times as large as the radius r.*
- *The volume needs to be at least V.*

To formulate an optimization problem, let $(r, h) \in \mathbb{R}^2$ be our decision variables. Assuming the aluminum sheet has a fixed thickness, the amount of aluminum used is determined by the surface area $f(r, h) = 2\pi r + 2\pi rh$, which will be our objective function. For the first requirement, we can write it as a constraint

$$h \geq 3r \iff -h + 3r \leq 0.$$

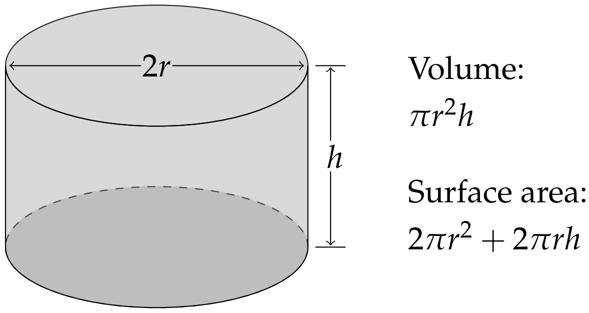

Figure 1: A cylindrical can of height h and radius r

Similarly, for the second requirement, we can write it as

$$h \leq 4r \iff h - 4r \leq 0.$$

The volume of this cylindrical can is $\pi r^2 h$*, so with the given parameter* V*, the last requirement can be written as*

$$V \leq \pi r^2 h \iff V - \pi r^2 h \leq 0.$$

While the variables r *and* h *must of course be nonnegative, we do not need to add bounds* $r \geq 0$ *and* $h \geq 0$ *because the first two constraints imply that* $4r \geq h \geq 3r$*, which guarantees* $r \geq 0$ *and hence also* $h \geq 3r \geq 0$*. To summarize, we have formulated the following optimization problem.*

$$\begin{aligned} \min \quad & 2\pi r^2 + 2\pi rh \\ \text{s.t.} \quad & -h + 3r \leq 0, \\ & h - 4r \leq 0, \\ & V - \pi r^2 h \leq 0, \\ & r, h \in \mathbb{R}. \end{aligned}$$

This is an example of continuous *and* nonlinear *optimization problem.*