

Simplex Method for Linear Optimization

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1 Standard Form of Linear Optimization (LO)

In graphical solutions for LO problems with two variables, we can find optimal solutions on vertices (the corner points). This observation can be more formally exploited by the *simplex method*. Before we present the algorithm, it is beneficial to consider *standard forms* of LO models, which have the following form.

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ \text{s. t.} \quad & \sum_{j=1}^n a_{ij} x_j = b_i, \quad i = 1, \dots, m, \\ & x_1, \dots, x_n \geq 0. \end{aligned} \tag{1}$$

Namely, we only have linear equality constraints and variable nonnegativity conditions in the standard form. Any LO formulation can be converted into the standard form by introducing auxiliary variables when necessary. For a less-than-or-equal-to constraint, we can use a *slack variable* $s \geq 0$ to rewrite it as an equality constraint

$$\sum_{i=1}^n a_i x_i \leq b \iff a_1 x_1 + \dots + a_n x_n + s = b, \quad s \geq 0. \tag{2}$$

Similarly, for the greater-than-or-equal-to constraint, we can use an *excess variable* $e \geq 0$ to rewrite it as an equality constraint

$$\sum_{i=1}^n a_i x_i \geq b \iff a_1 x_1 + \dots + a_n x_n - e = b, \quad e \geq 0. \tag{3}$$

For a *nonpositive variable*, we use its opposite to replace it in the decision variables

$$x_i \leq 0 \iff x'_i \geq 0. \tag{4}$$

If a variable is *free* (or *unrestricted in sign*), then we replace it with two nonnegative auxiliary variables

$$x_i \in \mathbb{R} \iff x_i = x'_i - x''_i, \quad x'_i, x''_i \geq 0. \quad (5)$$

Example 1. *The standard form of the following LO problem*

$$\begin{aligned} \max \quad & 3x_1 - 5x_2 + 7x_3 \\ \text{s. t.} \quad & 2x_1 + 4x_2 - x_3 \geq -3, \\ & 4x_1 - 2x_2 + 8x_3 \leq 7, \\ & 9x_1 + x_2 + 3x_3 = 11, \\ & x_1 \in \mathbb{R}, \\ & x_2 \leq 0, \\ & x_3 \geq 0. \end{aligned}$$

is given by

$$\begin{aligned} \max \quad & 3x'_1 - 3x''_1 + 5x'_2 + 7x_3 \\ \text{s. t.} \quad & 2x'_1 - 2x''_1 - 4x'_2 - x_3 - e_1 = -3, \\ & 4x'_1 - 4x''_1 + 2x'_2 + 8x_3 + s_2 = 7, \\ & 9x'_1 - 9x''_1 - x'_2 + 3x_3 = 11, \\ & x'_1, x''_1, x'_2, x_3, e_1, s_2 \geq 0. \end{aligned}$$

The standard form helps us to determine all variable values once we fix a subset of them to zero. To be precise, let N denote a subset of indices $\{1, \dots, n\}$ corresponding to *nonbasic* variables, and B a subset of indices of *basic* variables, such that the values of all basic variables can be determined through the equality constraints, once all nonbasic variables are fixed to zero. Sometimes B is called a *basis*. Any solution obtained by fixing nonbasic variables to zero is called a *basic solution*. The values of the basic variables in a basic solution could be negative. If all basic variables have nonnegative values, then the basic solution is further called a *basic feasible solution*.

Using linear algebra terminology, we can also say that B is an index set for basic variables when the submatrix A_B , that is formed by columns of A with indices in B , is nonsingular. Consequently, the number of basic variables should be the same as the number of constraints, assuming that the matrix A has full row rank. The geometric intuition of considering basic variables is that they can represent vertices of the feasible region. To see this, you may think about solving a system of m equations as finding the unique intersection point of m hyperplanes when the matrix A_B has full rank m .

Example 2. By renaming the variables in Example 1, we can write it as

$$\begin{array}{rcl}
 \max & 3x_1 & - 3x_2 + 5x_3 + 7x_4 \\
 \text{s. t.} & 2x_1 & - 2x_2 - 4x_3 - x_4 - x_5 = -3, \\
 & 4x_1 & - 4x_2 + 2x_3 + 8x_4 + x_6 = 7, \\
 & 9x_1 & - 9x_2 - x_3 + 3x_4 = 11, \\
 & & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.
 \end{array}$$

We can set $B = \{4, 5, 6\}$ and $N = \{1, 2, 3\}$. In this case, by setting $x_1 = x_2 = x_3 = 0$, the last constraint tells us that $x_4 = 11/3$, which then implies

$$\begin{aligned}
 x_5 &= 3 - x_4 = -\frac{2}{3}, \\
 x_6 &= 7 - 8x_4 = -\frac{67}{3}.
 \end{aligned}$$

Here, as $x_5, x_6 < 0$, we have an infeasible basic solution.

Example 3. Consider the following LO problem:

$$\begin{array}{rcl}
 \max & 5x_1 & + 5x_2 + 3x_3 \\
 \text{s. t.} & x_1 & + 3x_2 + x_3 \leq 3, \\
 & -x_1 & + 3x_3 \leq 2, \\
 & 2x_1 & - x_2 + 2x_3 \leq 4, \\
 & 2x_1 & + 3x_2 - x_3 \leq 2, \\
 & & x_1, x_2, x_3 \geq 0.
 \end{array}$$

We introduce slack variables $x_4, x_5, x_6, x_7 \geq 0$, the values of which can be uniquely determined by those of x_1, x_2, x_3 as

$$\begin{array}{rcl}
 z & = & 5x_1 + 5x_2 + 3x_3 \\
 x_4 & = & 3 - x_1 - 3x_2 - x_3, \\
 x_5 & = & 2 + x_1 - 3x_3, \\
 x_6 & = & 4 - 2x_1 + x_2 - 2x_3, \\
 x_7 & = & 2 - 2x_1 - 3x_2 + x_3.
 \end{array}$$

If we set $B = \{4, 5, 6, 7\}$ and $N = \{1, 2, 3\}$, then we can get a basic solution $x_1 = x_2 = x_3 = 0$, $x_4 = 3$, $x_5 = 2$, $x_6 = 4$, and $x_7 = 2$, which is feasible to the LO problem. The objective value, which we denote as z , is 0 at this basic feasible solution (bfs).

2 Simplex Method and Tableau

To move from a basic feasible solution to a “better” solution, we can select a nonbasic variable with positive “impact” on the objective value to become a basic variable. The new basic variable is called an *entering variable*, as it enters the basis, while the new nonbasic variable is called a *leaving variable*. We use Example 3 to illustrate the procedure as follows.

Example 3 (continued). *As both the nonbasic variables x_1 and x_2 have the largest coefficient 5 in the z-row, we pick x_1 to be the entering variable. To be more specific, we want to increase the value of x_1 until one of the basic variables x_4, x_5, x_6, x_7 reaches 0 (and thus becomes nonbasic), which is then the leaving variable.*

Iteration 1. We can do a ratio test based on the nonnegativity conditions:

$$\begin{aligned}x_4 &= 3 - x_1 \geq 0, \\x_5 &= 2 + x_1 \geq 0, \\x_6 &= 4 - 2x_1 \geq 0, \\x_7 &= 2 - 2x_1 \geq 0.\end{aligned}$$

The largest possible increase corresponds to the smallest ratio of the free coefficient to the absolute value of the coefficient for x_1 in the same row, assuming that the coefficient for x_1 is negative. As the ratios are 3, 2, 1 for variables x_4, x_6, x_7 , respectively, x_7 is then the leaving variable. We see that

$$x_1 = 1 - \frac{3}{2}x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_7.$$

By substituting this expression for x_1 in the other rows, we obtain the following new solution:

$$\begin{array}{r}z = 5 - \frac{5}{2}x_2 + \frac{11}{2}x_3 - \frac{5}{2}x_7 \\x_1 = 1 - \frac{3}{2}x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_7, \\x_4 = 2 - \frac{3}{2}x_2 - \frac{3}{2}x_3 + \frac{1}{2}x_7, \\x_5 = 3 - \frac{3}{2}x_2 - \frac{5}{2}x_3 - \frac{1}{2}x_7, \\x_6 = 2 + 4x_2 - 3x_3 + x_7.\end{array}$$

Here, the basic variables are indicated by $B = \{1, 4, 5, 6\}$ and nonbasic variables by $N = \{2, 3, 7\}$. Now since we still have one variable x_3 with positive coefficient in the z-row, we continue this procedure by setting it to be the entering variable.

Iteration 2. In the ratio test, we see that x_6 has the smallest ratio $\frac{2}{3}$ and thus should be the leaving variable. Using the relation

$$x_3 = \frac{2}{3} + \frac{4}{3}x_2 - \frac{1}{3}x_6 + \frac{1}{3}x_7,$$

we get

$$\begin{aligned} z &= \frac{26}{3} + \frac{29}{6}x_2 - \frac{2}{3}x_7 - \frac{11}{6}x_6 \\ x_3 &= \frac{2}{3} + \frac{4}{3}x_2 + \frac{1}{3}x_7 - \frac{1}{3}x_6, \\ x_1 &= \frac{4}{3} - \frac{5}{6}x_2 - \frac{1}{3}x_7 - \frac{1}{6}x_6, \\ x_4 &= 1 - \frac{7}{2}x_2 + \frac{1}{2}x_6, \\ x_5 &= \frac{4}{3} - \frac{29}{6}x_2 - \frac{4}{3}x_7 + \frac{5}{6}x_6. \end{aligned}$$

Now x_2 has a positive coefficient in the z -row, so we continue the procedure.

Iteration 3. We determine the leaving variable through the ratio test and find x_5 is the one with the smallest ratio $\frac{8}{29}$. By expressing x_2 through x_5, x_6, x_7 , we get

$$\begin{aligned} z &= 10 - 2x_7 - x_6 - x_5 \\ x_2 &= \frac{8}{29} - \frac{8}{29}x_7 + \frac{5}{29}x_6 - \frac{6}{29}x_5, \\ x_3 &= \frac{30}{29} - \frac{1}{29}x_7 - \frac{3}{29}x_6 - \frac{8}{29}x_5, \\ x_1 &= \frac{32}{29} - \frac{3}{29}x_7 - \frac{9}{29}x_6 + \frac{5}{29}x_5, \\ x_4 &= \frac{1}{29} + \frac{28}{29}x_7 - \frac{3}{29}x_6 + \frac{21}{29}x_5. \end{aligned}$$

All coefficients in the z -row are now negative. Hence, changing the value of any nonbasic variables will decrease the objective function value. We conclude that we have found an optimal solution for our LO problem, which has the value

$$x_1 = \frac{32}{29}, x_2 = \frac{8}{29}, x_3 = \frac{30}{29},$$

and all other variables being 0. The optimal objective value is $z = 10$.

The above computation procedure can be done more compactly in a *tableau* format. Without renaming the variables, we can also use BV_k and NV_k to indicate the basic and nonbasic variables in iteration k . We use the LO model from the baby carriers production problem as an example below.

Example 4. Consider the LO problem

$$\begin{aligned} \max \quad z &= 15x_1 + 25x_2 \\ \text{s. t.} \quad x_1 + x_2 + s_1 &= 450, \\ & \quad \quad \quad x_2 + s_2 = 300, \\ 4x_1 + 5x_2 + s_3 &= 2,000, \\ x_1 + s_4 &= 350, \\ x_1, x_2, s_1, s_2, s_3, s_4 &\geq 0. \end{aligned}$$

We can write it in a tableau format as follows.

z	x_1	x_2	s_1	s_2	s_3	s_4	rhs	$basis$
1	-15	-25	0	0	0	0	0	z
0	1	1	1	0	0	0	450	s_1
0	0	1	0	1	0	0	300	s_2
0	4	5	0	0	1	0	2,000	s_3
0	1	0	0	0	0	1	350	s_4

Here, rhs stands for right-hand side values, and the $basis$ stands for the basic variables corresponding to the rows. We use the convention that for the z -row, z is always in the basis and we revert the sign of the objective coefficients (by moving all variables in $z = 15x_1 + 25x_2$ to the left-hand side). In this initial setup, the basic feasible solution (bfs) consists of basic variables $BV_0 = \{s_1, s_2, s_3, s_4\}$ with values $s_1 = 450, s_2 = 300, s_3 = 2,000, s_4 = 350$, and nonbasic variables $NV_0 = \{x_1, x_2\}$ with values all being 0.

Iteration 1. To increase the value of z , we can increase the value of a nonbasic variable, which is called a pivot variable, and the corresponding column is called a pivot column in the tableau. As our objective function is linear, it is reasonable to select a nonbasic variable with the largest objective coefficient to be the pivot variable, which in this case is x_2 . As before, we conduct a ratio test to detect the pivot row as follows.

↓

z	x_1	x_2	s_1	s_2	s_3	s_4	rhs	$basis$	$ratio$
1	-15	-25	0	0	0	0	0	z	
0	1	1	1	0	0	0	450	s_1	450
0	0	1	0	1	0	0	300	s_2	300 ←
0	4	5	0	0	1	0	2,000	s_3	400
0	1	0	0	0	0	1	350	s_4	—

Now we can perform elementary row operations involving the pivot row with the goal of turning all pivot column entries in the non-pivot rows into 0 and the pivot row into 1. In our example, we multiply the pivot row by 25, -1 , and -5 , add the results to rows 0, 1, and 3, respectively.

z	x_1	x_2	s_1	s_2	s_3	s_4	rhs	$basis$
1	-15	0	0	25	0	0	7,500	z
0	1	0	1	-1	0	0	150	s_1
0	0	1	0	1	0	0	300	x_2
0	4	0	0	-5	1	0	500	s_3
0	1	0	0	0	0	1	350	s_4

After the first iteration, the bfs consists of $BV_1 = \{s_1, x_2, s_3, s_4\}$ with values $s_1 = 150, x_2 =$

300, $s_3 = 500$, $s_4 = 350$, with $NV_1 = \{x_1, s_2\}$ with values 0. The objective value $z = 7,500$.
 Iteration 2. Now the variable x_1 has the largest coefficient in the objective, so we repeat the ratio test and find the pivot row.

$$\downarrow$$

z	x_1	x_2	s_1	s_2	s_3	s_4	rhs	$basis$	$ratio$
1	-15	0	0	25	0	0	7,500	z	
0	1	0	1	-1	0	0	150	s_1	150
0	0	1	0	1	0	0	300	x_2	—
0	4	0	0	-5	1	0	500	s_3	125 ←
0	1	0	0	0	0	1	350	s_4	350

We perform the elementary row operations and get the updated tableau as follows.

z	x_1	x_2	s_1	s_2	s_3	s_4	rhs	$basis$
1	0	0	0	25/4	15/4	0	9,375	z
0	0	0	1	1/4	-1/4	0	25	s_1
0	0	1	0	1	0	0	300	x_2
0	1	0	0	-5/4	1/4	0	125	x_1
0	0	0	0	5/4	-1/4	1	225	s_4

Now the bfs consists of $BV_2 = \{s_1, x_2, x_1, s_4\}$ with values $s_1 = 25$, $x_2 = 300$, $x_1 = 125$, $s_4 = 225$, and $NV_2 = \{s_2, s_3\}$ with values 0. The objective value $z = 9,375$. Note that increasing the value of any nonbasic variable would now decrease the objective value, which can be seen from the nonnegativity of the coefficients in the z -row of the tableau. We have thus found an optimal solution to the LO problem. From Figure 1, we see that the above algorithmic procedure corresponds to moving from point $(0,0)$ to $(0,300)$, and then to $(125,300)$ in the (x_1, x_2) -plane.

Remark. In a simplex tableau (of a LO maximization problem), the basic feasible solution is optimal if all nonbasic variables have nonnegative coefficients in the z -row.

Recall that a LO problem can be unbounded. We would like to detect unboundedness from the simplex tableau as described in the next example.

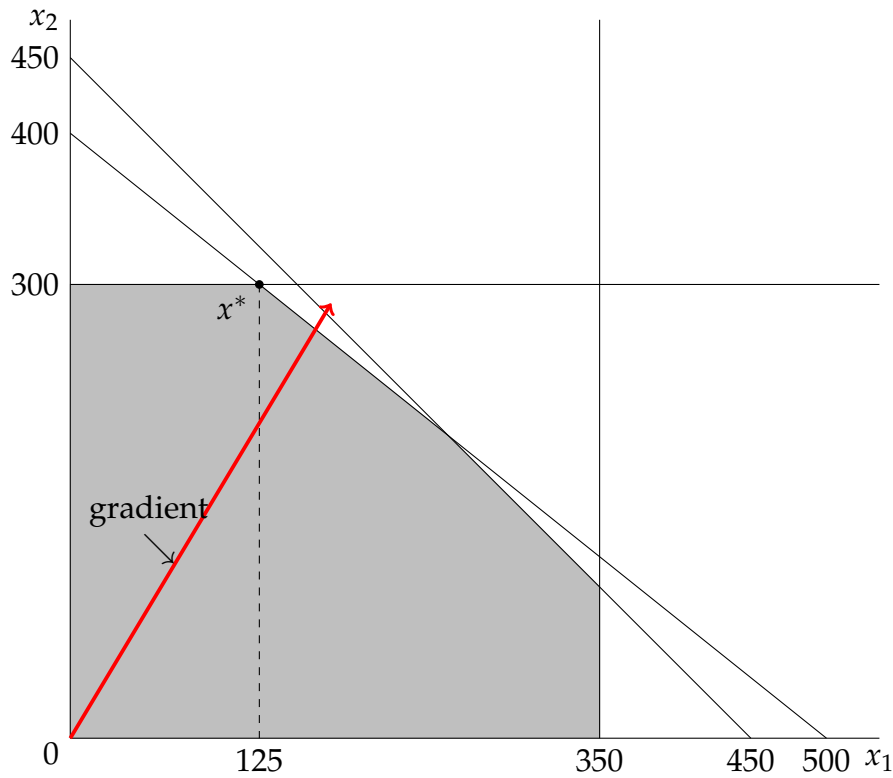


Figure 1: Feasible region and objective direction of Example 4

Example 5. Consider the following tableau in a LO maximization problem.

	↓								
z	x_1	x_2	s_1	s_2	s_3	s_4	rhs	$basis$	
1	25	-4	0	0	0	0	90	z	
0	14	-1	1	0	0	0	25	s_1	
0	1	0	0	1	0	0	30	s_2	
0	-5	-14	0	0	1	0	12	s_3	
0	4	-7	0	0	0	1	22	s_4	

We see that by increasing the value of the nonbasic variable x_2 , the objective value should increase. However, there is no restriction on how much x_2 can increase (as no ratio test needs to be performed). This means that this LO problem is unbounded.

Remark. In a simplex tableau (of a LO maximization problem), the unboundedness is detected if there is a column with no positive entries.

3 Simplex Method Termination and Initialization

A natural question is whether simplex method can always find an optimal solution in finitely many steps/iterations. To answer this question, we need to note a special situa-

tion called *degeneracy*, where there is one or more basic variables equal to 0. Degeneracy (or *degenerate bfs*) may lead to the *cycling* phenomenon as in the next example.

Example 6. Consider the LO problem

$$\begin{aligned} \max \quad & 5x_1 + 4x_2 - 20x_3 - 2x_4 \\ \text{s. t.} \quad & \frac{1}{4}x_1 - \frac{1}{8}x_2 + 12x_3 + 10x_4 \leq 0, \\ & \frac{1}{10}x_1 + \frac{1}{20}x_2 + \frac{1}{20}x_3 + \frac{1}{5}x_4 \leq 0, \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

The initial (iteration 0) tableau can be written as follows.

z	x_1	x_2	x_3	x_4	x_5	x_6	rhs	$basis$
1	-5	-4	20	2	0	0	0	z
0	$\boxed{\frac{1}{4}}$	$-\frac{1}{8}$	12	10	1	0	0	x_5
0	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{5}$	0	1	0	x_6

We continue the simplex method.

• Iteration 1

z	x_1	x_2	x_3	x_4	x_5	x_6	rhs	$basis$
1	0	$-\frac{13}{2}$	260	202	20	0	0	z
0	1	$-\frac{1}{2}$	48	40	4	0	0	x_1
0	0	$\boxed{\frac{1}{10}}$	$-\frac{19}{4}$	$-\frac{19}{5}$	$-\frac{2}{5}$	1	0	x_6

• Iteration 2

z	x_1	x_2	x_3	x_4	x_5	x_6	rhs	$basis$
1	0	0	$-\frac{195}{4}$	-45	-6	65	0	z
0	1	0	$\boxed{\frac{97}{4}}$	21	2	5	0	x_1
0	0	1	$-\frac{95}{2}$	-38	-4	10	0	x_2

• Iteration 3

z	x_1	x_2	x_3	x_4	x_5	x_6	rhs	$basis$
1	$\frac{195}{97}$	0	0	$-\frac{270}{97}$	$-\frac{192}{97}$	$\frac{7280}{97}$	0	z
0	$\frac{190}{97}$	1	0	$\boxed{\frac{304}{97}}$	$-\frac{8}{97}$	$\frac{1920}{97}$	0	x_2
0	$\frac{4}{97}$	0	1	$\frac{84}{97}$	$\frac{8}{97}$	$\frac{20}{97}$	0	x_3

• Iteration 4

z	x_1	x_2	x_3	x_4	x_5	x_6	rhs	$basis$
1	$\frac{15}{4}$	$\frac{135}{152}$	0	0	$-\frac{39}{19}$	$\frac{1760}{19}$	0	z
0	$-\frac{1}{2}$	$-\frac{21}{76}$	1	0	$\frac{2}{19}$	$-\frac{100}{19}$	0	x_3
0	$\frac{5}{8}$	$\frac{97}{304}$	0	1	$-\frac{1}{38}$	$\frac{120}{19}$	0	x_4

• Iteration 5

z	x_1	x_2	x_3	x_4	x_5	x_6	rhs	$basis$
1	-6	$-\frac{9}{2}$	$\frac{39}{2}$	0	0	-10	0	z
0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	1	0	5	0	x_4
0	$-\frac{19}{4}$	$-\frac{21}{8}$	$\frac{19}{2}$	0	1	-50	0	x_5

• Iteration 6

z	x_1	x_2	x_3	x_4	x_5	x_6	rhs	$basis$
1	-5	-4	20	2	0	0	0	z
0	$\frac{1}{4}$	$-\frac{1}{8}$	12	10	1	0	0	x_5
0	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{5}$	0	1	0	x_6

Note that in iteration 6, we get the same tableau as we got in iteration 0. Thus if we continue with the execution of the simplex method, we will keep repeating the calculations in iterations 0-6 and will never be able to leave the same solution.

To avoid cycling in the simplex method, we can use certain *pivoting rule*, which determines the pivoting variable at every degenerate solution. A conceptually useful rule is called *Bland’s rule*: assuming that the variables are indexed, for example, by $1, \dots, n$, we always choose the entering or leaving variable with the smallest index. We illustrate Bland’s rule by applying it to Example 6.

Example 6 (continued). *Applying Bland’s rule leads to the same first 5 iterations, at the end of which we have the following tableau.*

z	x_1	x_2	x_3	x_4	x_5	x_6	rhs	$basis$
1	-6	$-\frac{9}{2}$	$\frac{39}{2}$	0	0	-10	0	z
0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	1	0	5	0	x_4
0	$-\frac{19}{4}$	$-\frac{21}{8}$	$\frac{19}{2}$	0	1	-50	0	x_5

Now the candidates for the entering variables are $x_1, x_2,$ and x_6 , so by Bland’s rule we choose

x_1 to enter the basis.

z	x_1	x_2	x_3	x_4	x_5	x_6	rhs	$basis$
1	0	$-\frac{3}{2}$	$\frac{45}{2}$	12	0	50	0	z
0	1	$\frac{1}{2}$	$\frac{1}{2}$	2	0	10	0	x_1
0	0	$-\frac{1}{4}$	$\frac{95}{8}$	$\frac{19}{2}$	1	$-\frac{5}{2}$	0	x_5

After one more iteration, we see optimality from the tableau.

z	x_1	x_2	x_3	x_4	x_5	x_6	rhs	$basis$
1	3	0	24	18	0	80	0	z
0	2	1	1	4	0	20	0	x_2
0	$\frac{1}{2}$	0	$\frac{97}{8}$	$\frac{21}{2}$	1	$\frac{5}{2}$	0	x_5

In fact, the solution value is the same as the initial bfs, but the last tableau shows its optimality.

It can be shown that with Bland's rule, simplex method does not have the cycling phenomenon. The proof is a little involved, so we do not go into the details here. In practice, Bland's rule may not lead to efficient implementation of the simplex method. Other pivoting rules may be more favorable, such as the *steepest edge rule*, the *random edge rule*, or the *lexicographic rule*. The details can be found in [1].

Without cycling, the simplex method is guaranteed to terminate in finitely many iteration (with either an optimal bfs or a certificate for unboundedness). In fact, given a standard LO form with n variables and m constraints, there are at most $\binom{n}{m}$ possible simplex tableaus. This is because each tableau has a unique set of basic variables, the cardinality (i.e., size) of which is exactly m . By definition, we will not see the same basis twice unless cycling happens.

Theorem 1. *If the simplex method does not have cycling, then it terminates with at most $\binom{n}{m}$ iterations.*

Another issue is regarding how to find feasible solutions to the LO problem. We have seen that if we start with a bfs, then the simplex method can keep feasibility through the ratio tests. Therefore, it suffices for us to discuss initialization procedures for feasible solutions. Here, we describe two methods: the *two-phase* simplex method and the *big-M* simplex method.

The main idea of both methods is to "relax" the constraints by introducing artificial variables, and then try to find solutions where the artificial variables are all zero. To be

precise, for a standard form LO problem

$$\begin{aligned}
 \max \quad & \sum_{i=1}^n c_i x_i \\
 \text{s. t.} \quad & \sum_{i=1}^n a_{ji} x_i = b_j, \quad j = 1, \dots, m, \\
 & x_i \geq 0, \quad i = 1, \dots, n,
 \end{aligned} \tag{6}$$

we define the following index subsets of $J := \{1, \dots, m\}$: $J^+ := \{j \in J : b_j \geq 0\}$, and $J^- := \{j \in J : b_j < 0\}$, representing the constraints that have positive or no violations, and those that have negative violations if we set $x = 0$. Then the two-phase method first solves the first-phase feasibility problem

$$\begin{aligned}
 \min \quad & \sum_{j \in J} u_j \\
 \text{s. t.} \quad & \sum_{i=1}^n a_{ji} x_i + u_j = b_j, \quad \forall j \in J^+, \\
 & \sum_{i=1}^n a_{ji} x_i - u_j = b_j, \quad \forall j \in J^-, \\
 & x_i \geq 0, \quad i = 1, \dots, n, \\
 & u_j \geq 0, \quad j = 1, \dots, m.
 \end{aligned} \tag{7}$$

Clearly, if we set $x = 0$ and $u_j = |b_j|$ for $j = 1, \dots, m$, then (x, u) is a bfs to the problem (7). Moreover, the original problem (6) is feasible if and only if (7) has an optimal basic solution (x', u') where u'_1, \dots, u'_m are all nonbasic variables. In this case, the objective value of (7) will also be 0, meaning that we do not need these artificial variables to make the original problem (6) feasible. Then we can use the solution x' as a bfs to (6). In practice, if an equality constraint j in (6) is converted from an inequality constraint, then we can directly take the artificial variable u_j to be the slack or excess variable. We illustrate the two-phase simplex method by the following example.

Example 7. Consider the following LO problem:

$$\begin{aligned}
 \max \quad & 5x_1 + 10x_2 \\
 \text{s. t.} \quad & 2x_1 + x_2 = 4 \\
 & x_1 + 2x_2 \leq 5 \\
 & x_1, x_2 \geq 0.
 \end{aligned}$$

We can convert into the standard form (by introducing the slack variable $s_2 \geq 0$) and then

construct the first-phase feasibility problem as

$$\begin{aligned} \max \quad & -a_1 \\ \text{s. t.} \quad & 2x_1 + x_2 + a_1 = 4 \\ & x_1 + 2x_2 + s_2 = 5 \\ & x_1, x_2, s_2, a_1 \geq 0. \end{aligned}$$

We start with the bfs $(x_1, x_2, s_2, a_1) = (0, 0, 5, 4)$ and use the simplex method as follows.

- Iteration 0 (raw)

z	x_1	x_2	a_1	s_2	rhs	basis
1	0	0	1	0	0	z
0	2	1	1	0	4	a_1
0	1	2	0	1	5	s_2

As a basic variable, a_1 should not have a nonzero coefficient in the z-row. We need to get a correct tableau by a row operation.

- Iteration 0

z	x_1	x_2	a_1	s_2	rhs	basis
1	-2	-1	0	0	-4	z
0	2	1	1	0	4	a_1
0	1	2	0	1	5	s_2

- Iteration 1

z	x_1	x_2	a_1	s_2	rhs	basis
1	0	0	1	0	0	z
0	1	1/2	1/2	0	2	x_1
0	0	3/2	-1/2	1	3	s_2

The tableau is optimal and we can check that $(x_1, x_2, s_2) = (2, 0, 3)$ is a bfs of the original problem. Thus the first-phase feasibility problem is solved.

We start the second phase as follows.

- Iteration 0 (raw)

z	x_1	x_2	s_2	rhs	basis
1	-5	-10	0	0	z
0	1	1/2	0	2	x_1
0	0	3/2	1	3	s_2

As x_1 is a basic variable, its coefficient in the z-row must be zero. We can correct this again by a row operation.

- Iteration 0

z	x_1	x_2	s_2	rhs	$basis$
1	0	$-15/2$	0	10	z
0	1	$1/2$	0	2	x_1
0	0	$3/2$	1	3	s_2

The entering variable is x_2 and the ratio test determines that s_2 should be leaving.

- Iteration 1

z	x_1	x_2	s_2	rhs	$basis$
1	0	0	5	25	z
0	1	0	$-1/3$	1	x_1
0	0	1	$2/3$	2	x_2

The tableau is optimal. We have solved the second phase problem.

From the last tableau, we can see that an optimal solution is $(x_1^*, x_2^*) = (1, 2)$, with the optimal value $z^* = 25$.

The big-M method defines a big coefficient $M \gg 0$ for the artificial variables in the objective function. To be precise, for the problem (6), we pick a large constant $M > 0$ and solve the following problem

$$\begin{aligned}
 \max \quad & \sum_{i=1}^n c_i x_i - M \sum_{j=1}^m u_j \\
 \text{s. t.} \quad & \sum_{i=1}^n a_{ji} x_i + u_j = b_j, \quad \forall j \in J^+, \\
 & \sum_{i=1}^n a_{ji} x_i - u_j = b_j, \quad \forall j \in J^-, \\
 & x_i \geq 0, \quad i = 1, \dots, n, \\
 & u_j \geq 0, \quad j = 1, \dots, m.
 \end{aligned} \tag{8}$$

The subsets J^+ and J^- are defined similarly as in the two-phase method, so we can choose the obvious bfs $x_i = 0$ for $i = 1, \dots, n$ and $u_j = |b_j|$ for $j = 1, \dots, m$, as our starting point. We say that M is sufficiently large, if in the simplex iterations, the sign of any linear expression involving M would only depend on the coefficient of M . For example, $-M + 10 < 0$ as M has a coefficient of -1 , and $2M + 30 > M + 100$ as the coefficient of M on the left-hand side is 2, which is greater than 1, the coefficient of M on the right-hand side. It can be shown that assuming M is sufficiently large,

- if the problem (8) has an optimal solution (x^*, u^*) with $u^* = 0$, then x^* is an optimal solution to (6);
- if the problem (8) has an optimal solution (x^*, u^*) with $u^* \neq 0$, (6) is infeasible;

- if the problem (8) is unbounded, then so is (6).

We illustrate below the big-M method on the same problem that appeared in Example 7.

Example 8. For the problem

$$\begin{aligned} \max \quad & 5x_1 + 10x_2 \\ \text{s. t.} \quad & 2x_1 + x_2 = 4 \\ & x_1 + 2x_2 \leq 5 \\ & x_1, x_2 \geq 0, \end{aligned}$$

the big-M formulation is

$$\begin{aligned} \max \quad & 5x_1 + 10x_2 - Ma_1 \\ \text{s. t.} \quad & 2x_1 + x_2 + a_1 = 4 \\ & x_1 + 2x_2 + s_2 = 5 \\ & x_1, x_2, s_2, a_1 \geq 0. \end{aligned}$$

- Iteration 0 (raw)

z	x_1	x_2	a_1	s_2	rhs	basis
1	-5	-10	M	0	0	z
0	2	1	1	0	4	a_1
0	1	2	0	1	5	s_2

We need to eliminate the coefficients of basic variables in the z-row through row operations.

- Iteration 0

z	x_1	x_2	a_1	s_2	rhs	basis
1	$-5 - 2M$	$-10 - M$	0	0	$-4M$	z
0	2	1	1	0	4	a_1
0	1	2	0	1	5	s_2

Here, x_1 should be the entering variable because $-5 - 2M < -10 - M$ for sufficiently large M . By the ratio test, a_1 should be leaving.

- Iteration 1

z	x_1	x_2	a_1	s_2	rhs	basis
1	0	$-15/2$	$5/2 + M$	0	10	z
0	1	$1/2$	$1/2$	0	2	x_1
0	0	$3/2$	$-1/2$	1	3	s_2

Now x_2 enters and s_2 leaves the basis.

- Iteration 2

z	x_1	x_2	a_1	s_2	rhs	$basis$
1	0	0	M	5	25	z
0	1	0	$2/3$	$-1/3$	1	x_1
0	0	1	$-1/3$	$2/3$	2	x_2

We terminate the simplex method as all coefficients in the z -row are nonnegative. We have found an optimal solution $(x_1^*, x_2^*) = (1, 2)$. It is feasible to the original problem because $a_1^* = 0$ is nonbasic. The optimal value is $z^* = 25$.

4 Simplex Method in the Matrix Form

It may be conceptually simpler to consider the simplex method in the following matrix form. The standard form can be written as

$$\begin{aligned} \max \quad & c^T x \\ \text{s. t.} \quad & Ax = b, \\ & x \geq 0. \end{aligned} \tag{9}$$

By using the index sets B (for basic variables) and N (for nonbasic variables), we use the subscript notation to indicate the data associated with these variables. For example, if $B = \{1, 3, 4\}$, then $c_B = (c_1, c_3, c_4)$ is a vector consisting of components of the vector c with indices in B . Thus given B and N , the standard form can be rewritten as

$$\begin{aligned} \max \quad & [c_B^T, c_N^T] \begin{bmatrix} x_B \\ x_N \end{bmatrix} \\ \text{s. t.} \quad & [A_B, A_N] \begin{bmatrix} x_B \\ x_N \end{bmatrix} = b, \\ & \begin{bmatrix} x_B \\ x_N \end{bmatrix} \geq 0. \end{aligned} \iff \begin{aligned} \max \quad & c_B^T x_B + c_N^T x_N \\ \text{s. t.} \quad & A_B x_B + A_N x_N = b, \\ & x_B \geq 0, x_N \geq 0. \end{aligned} \tag{10}$$

Then each iteration of the simplex method can be written in the *matrix form*:

$$\begin{aligned} z &= c_B^T A_B^{-1} b + r^T x_N \\ x_B &= p + Q x_N \end{aligned} \tag{11}$$

where $p = A_B^{-1} b$, $Q = [q_{ij}] = -A_B^{-1} A_N$, and the vector $r := c_N - (c_B^T A_B^{-1} A_N)^T$ is sometimes called the *reduced cost*. Recall that the definition of a basis requires the submatrix A_B to be nonsingular, which ensures that the inverse A_B^{-1} is well-defined. Equivalently,

we can write the tableau in each iteration as

$$\begin{array}{c|cccc}
 z & x_B & x_N & rhs \\
 \hline
 1 & 0 & -r & c_B^T A_B^{-1} b \\
 0 & I & -Q & p \\
 \hline
 \end{array} \tag{12}$$

where I in the tableau is the m -by- m identity matrix.

The optimality criterion for the simplex method (for a LO maximization problem) can be restated as $r \leq 0$, and if any $r_k > 0$, then x_k should be the entering variable. The leaving variable is determined by the ratio test, i.e., any x_j such that

$$q_{jk} < 0 \quad \text{and} \quad -\frac{p_j}{q_{jk}} = \min \left\{ -\frac{p_i}{q_{ik}} : q_{ik} < 0, i = 1, \dots, m \right\}. \tag{13}$$

The unboundedness can be detected if for all $i = 1, \dots, m$, $q_{ik} \geq 0$. Otherwise, we set $B \leftarrow B \cup \{j\} \setminus \{k\}$ and continue. We use the data in Example 3 to illustrate the procedure in the matrix form.

Example 9. We write the standard form of the LO problem

$$\begin{array}{ll}
 \max & 5x_1 + 5x_2 + 3x_3 \\
 \text{s. t.} & x_1 + 3x_2 + x_3 \leq 3, \\
 & -x_1 + 3x_3 \leq 2, \\
 & 2x_1 - x_2 + 2x_3 \leq 4, \\
 & 2x_1 + 3x_2 - x_3 \leq 2, \\
 & x_1, x_2, x_3 \geq 0.
 \end{array}$$

with slack variables $x_4, x_5, x_6, x_7 \geq 0$ through the matrix and the vectors

$$A = \begin{bmatrix} 1 & 3 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 3 & 0 & 1 & 0 & 0 \\ 2 & -1 & 2 & 0 & 0 & 1 & 0 \\ 2 & 3 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 2 \\ 4 \\ 2 \end{bmatrix}, \quad c = \begin{bmatrix} 5 \\ 5 \\ 3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Initially, we set $B \leftarrow B_0 = \{4,5,6,7\}$ and $N \leftarrow N_0 = \{1,2,3\}$, then we have

$$A_B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_N = \begin{bmatrix} 1 & 3 & 1 \\ -1 & 0 & 3 \\ 2 & -1 & 2 \\ 2 & 3 & -1 \end{bmatrix}, \quad c_B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad c_N = \begin{bmatrix} 5 \\ 5 \\ 3 \end{bmatrix}.$$

In particular,

$$p = A_B^{-1}b = \begin{bmatrix} 3 \\ 2 \\ 4 \\ 2 \end{bmatrix}, \quad Q = -A_B^{-1}A_N = \begin{bmatrix} -1 & -3 & -1 \\ 1 & 0 & -3 \\ -2 & 1 & -2 \\ -2 & -3 & 1 \end{bmatrix}, \quad r = c_N - (c_B^T A_B^{-1} A_N)^T = \begin{bmatrix} 5 \\ 5 \\ 3 \end{bmatrix}.$$

Iteration 1. We pick x_1 as an entering variable, and the ratio test determines that x_7 is a leaving variable, Thus $B_1 = B_0 \cup \{1\} \setminus \{7\} = \{1,4,5,6\}$, and consequently $N_1 = \{2,3,7\}$. Now setting $B \leftarrow B_1$ and $N \leftarrow N_1$, we get

$$p = A_B^{-1}b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \end{bmatrix}, \quad Q = -A_B^{-1}A_N = \begin{bmatrix} -\frac{3}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{3}{2} & -\frac{3}{2} & \frac{1}{2} \\ -\frac{3}{2} & -\frac{5}{2} & -\frac{1}{2} \\ 4 & -3 & 1 \end{bmatrix}, \quad r = c_N - (c_B^T A_B^{-1} A_N)^T = \begin{bmatrix} -\frac{5}{2} \\ \frac{11}{2} \\ -\frac{5}{2} \end{bmatrix}.$$

Iteration 2. Now x_3 is the new entering variable, and by the ratio test, x_6 is leaving. Thus $B_2 = B_1 \cup \{3\} \setminus \{6\} = \{1,3,4,5\}$, and consequently $N_2 = \{2,6,7\}$. Now setting $B \leftarrow B_2$ and $N \leftarrow N_2$, we get

$$p = A_B^{-1}b = \begin{bmatrix} \frac{2}{3} \\ \frac{4}{3} \\ 1 \\ \frac{4}{3} \end{bmatrix}, \quad Q = -A_B^{-1}A_N = \begin{bmatrix} \frac{4}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{5}{6} & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{7}{2} & +\frac{1}{2} & 0 \\ -\frac{29}{6} & \frac{5}{6} & -\frac{4}{3} \end{bmatrix}, \quad r = c_N - (c_B^T A_B^{-1} A_N)^T = \begin{bmatrix} \frac{29}{6} \\ -\frac{11}{6} \\ -\frac{2}{3} \end{bmatrix}.$$

Iteration 3. The variable x_2 is entering and by the ratio test x_5 is leaving. Thus $B_3 = B_2 \cup \{2\} \setminus \{5\} = \{1,2,3,4\}$, and consequently $N_3 = \{5,6,7\}$. Now setting $B \leftarrow B_3$ and $N \leftarrow N_3$, we get

$$p = A_B^{-1}b = \begin{bmatrix} \frac{8}{29} \\ \frac{30}{29} \\ \frac{32}{29} \\ \frac{1}{29} \end{bmatrix}, \quad Q = -A_B^{-1}A_N = \begin{bmatrix} -\frac{6}{29} & \frac{5}{29} & -\frac{8}{29} \\ -\frac{8}{29} & -\frac{3}{29} & -\frac{1}{29} \\ \frac{5}{29} & -\frac{9}{29} & -\frac{3}{29} \\ \frac{21}{29} & -\frac{3}{29} & \frac{28}{29} \end{bmatrix}, \quad r = c_N - (c_B^T A_B^{-1} A_N)^T = \begin{bmatrix} -1 \\ -1 \\ -2 \end{bmatrix}.$$

As $r \leq 0$, we have found an optimal solution $x^* = (32/29, 8/29, 30/29, 0, 0, 0, 0)$ with the

optimal value $z^* = 10$.

References

- [1] Matoušek, Jiří, and Bernd Gärtner. Understanding and using linear programming. Vol. 1. Berlin: Springer, 2007.